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Evaporation from a cylindrical surface into vacuum

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When evaporation takes place in a surrounding vacuum, the expanding flow from a cylindrical surface is expected to start subsonic and to become supersonic in a short distance. A detailed treatment of this transition is given based on moment equations derived from the BGK model equation using an ellipsoidal approximant to the distribution function. Asymptotic solutions are developed for large source Reynolds numbers and compared with previous treatments. For moderate source Reynolds numbers a numerical procedure is used. In the latter case the treatment predicts that the flow never approaches a state of translational equilibrium.

1. Introduction

This study is motivated by interest in evaporation from a line source of finite width into vacuum. The distinction between evaporation and effusion is basically that in the former case the Mach number at the source is not controlled by gas-phase phenomena (i.e. sound waves) and is generally expected to be subsonic. Evaporation from a source of finite width generally exhibits two-dimensional effects in the gas phase, as the flow contracts to become supersonic, and, if the source is a liquid, owing to surface distortion caused by momentum recoil from the evaporating vapour. There can also be unsteady effects. Here we shall treat only steady one-dimensional evaporation from the surface of an infinitely long circular cylinder of constant diameter. Only a monatomic gas is considered, and surface recession is overlooked. The gas is assumed to expand to zero back-pressure far from the surface. Evaporation rates considered, defined later in terms of a non-dimensional source Reynolds number, range from moderate to large.

Previous treatments of expansion of a source flow into vacuum by Edwards & Cheng (1966) and Hamel & Willis (1966) are based on using hypersonic approximations to the moment equations. They predict that translational equilibrium breaks down as the flow becomes increasingly rarefied. Such breakdown is observed experimentally. (Cf. Anderson & Fenn 1965.) Hamel & Willis (1966) relate this far-field solution to the supersonic near field by asymptotically matching it to an equilibrium (isentropic) flow in some transition regime.

When we consider moderate evaporation rates this sort of approach needs to be re-evaluated for two reasons.

(i) Deviation from isentropic supersonic flow begins at a Mach number which decreases with the evaporation rate. Thus non-equilibrium behaviour can occur at moderate, not hypersonic, flow speeds, and moment equations based on the hypersonic approximation can be inappropriate.

(ii) The vapour flow is expected to start subsonic at the surface, but it is intuitively reasonable that it should become supersonic. Acceleration of the subsonic flow can occur only with entropy production, and hence the flow near the surface is generally not near translational equilibrium.

Taken together these imply that the flow may never approach a state of translational equilibrium for sufficiently low evaporation rates. Also, a more detailed examination of the Knudsen layer between the surface and the sonic station can be relevant.

Edwards & Collins (1969) analysed transition from subsonic to supersonic flow around a spherical drop evaporating into vacuum on the basis of the Navier–Stokes equations. Their analysis is limited to large source Reynolds numbers and shows that viscous effects do provide a viable mechanism for such transition. The usual question of validity of the Navier–Stokes equations in strong translational equilibrium arises.

A Mott-Smith treatment of the Knudsen layer was used by Anisimov (1968). The analysis presented later in this paper indicates that Anisimov's results are limited to large source Reynolds number. There is general agreement between the predictions of Edwards & Collins (1969) and Anisimov (1968).

An independent approach will be taken here which is based on the BGK model equation. Instead of using matched asymptotic expansions, an approximant to the distribution function is introduced which appears to have enough structure to give a good estimate in all flow regimes. Specifically,

$$f \sim \rho(2\pi R)^{-\frac{3}{2}}(T_r T_\theta T_z)^{-\frac{1}{2}} \exp\left[-\frac{(\xi-u)^2}{2RT_r}-\frac{\eta^2}{2RT_\theta}-\frac{\zeta^2}{2RT_z}\right], \tag{1.1}$$

where ρ is the gas density, $R = k/m$ is the gas constant, u is the mean radial velocity, and ξ, η, ζ and T_r, T_θ, T_z are the velocities and temperatures in the radial, polar and axial directions, respectively.

This ellipsoidal form is consistent with the moment equations of Edwards & Cheng (1966) and Hamel & Willis (1966), and it obviously reduces to the Maxwellian form if the three temperatures are equal. That use of (1.1) entails error was predicted by Hamel, Willis & Lin (1972) and recently supported by Cattolica, Robben & Talbot (1974). Nonetheless, here it will be used.

The ellipsoidal form has been used by Holway (1964) in studying shock structure. He found that his results reduced to the Navier–Stokes prediction for weak shock waves and were similar to the Mott-Smith prediction for strong shocks.

2. Five-moment BGK model equations

There are five quantities in the ellipsoidal approximant to be determined by appropriate moment equations: ρ, u, T_r, T_θ, and T_z. Here these moment equations will be derived from the BGK model, which in cylindrical symmetry is

$$\frac{\partial}{\partial r}(r\xi f)+\frac{\partial}{\partial \xi}(\eta^2 f)-\frac{\partial}{\partial \eta}(\xi\eta f) = \nu r(F-f), \tag{2.1}$$

where

$$F = \rho(2\pi RT)^{-\frac{3}{2}}\exp\left[-\frac{(\xi-u)^2+\eta^2+\zeta^2}{2RT}\right] \tag{2.2}$$

is the equilibrium distribution function and the mean temperature

$$T = \tfrac{1}{3}(T_r+T_\theta+T_z).$$

Definitions of the various parameters are given after (1.1).

Taking moments of (2.1) with respect to 1, ξ, ξ^2, ξ^3, $\xi\eta^2$ and $\xi\zeta^2$ leads to the following equations:

$$\frac{d}{dr}(\rho ur) = 0, \tag{2.3a}$$

$$\frac{u}{R}\frac{du}{dr}+\frac{1}{\rho}\frac{d}{dr}(\rho T_r) = \frac{T_\theta-T_r}{r}, \tag{2.3b}$$

$$3\frac{dT_r}{dr}+\frac{2u}{R}\frac{du}{dr} = \frac{\nu}{u}(T-T_r)+\frac{2T_\theta}{r}, \tag{2.3c}$$

$$\frac{dT_\theta}{dr} = \frac{\nu}{u}(T-T_\theta)-\frac{2T_\theta}{r}, \tag{2.3d}$$

$$\frac{dT_z}{dr} = \frac{\nu}{u}(T-T_z). \tag{2.3e}$$

For the special case of Maxwellian molecules precisely the same moments result from the Boltzmann equation provided the collision frequency $\nu = \rho RT/\mu(T)$, where μ is the first coefficient of viscosity. This form for the collision frequency is taken to apply generally, and a power-law viscosity is assumed: $\mu = \mu_s(T/T_s)^{1-\beta}$ with $0 \leqslant \beta \leqslant \frac{1}{2}$.

An interesting consequence of these moment equations can be derived. Adding (2.3*c*–*e*) gives $d(3T_r+T_\theta+T_z+u^2R^{-1})/dr = 0$, or, recalling that

$$T = \tfrac{1}{3}(T_r+T_\theta+T_z),$$

$$\tfrac{3}{2}RT+RT_r+\tfrac{1}{2}u^2 = \text{constant}. \tag{2.4}$$

Thus in this treatment a generalized form of stagnation enthalpy is preserved throughout the gas flow, independently of the collision-frequency model used. Note, in particular, that the flow work contribution is RT_r as would be expected since there is directed movement only in the radial direction.

Entropy is generally not preserved in the gas flow. A general expression for the entropy per unit mass is given by Vincenti & Kruger (1967) as

$$s = \frac{R}{\rho}\int\int\int_{-\infty}^{\infty} f\left[1+\ln\left(\frac{m^4}{h^3 f}\right)\right]d\xi\, d\eta\, d\zeta, \tag{2.5}$$

where f is the distribution function as used here, m is the mass of an atom and h is Planck's constant. Substitution of the ellipsoidal approximant (1.1) gives

$$s/R = \tfrac{5}{2}+\ln\left[\frac{m^4}{h^3}\frac{1}{\rho}((2\pi R)^3\, T_r T_\theta T_z)^{\frac{1}{2}}\right]. \tag{2.6}$$

From the moment equations (2.3) it follows that

$$\frac{r}{R}\frac{ds}{dr} = \frac{\nu r}{2u}\left(\frac{T}{T_r}+\frac{T}{T_\theta}+\frac{T}{T_z}-3\right). \tag{2.7}$$

One circumstance in which entropy is preserved is when all temperatures are equal.

For later use it will be worthwhile solving (2.3) for all first derivatives. The results will be stated in terms of the non-dimensional variables $y = \ln(r/r_s)$, $\tilde{\rho} = \rho/\rho_s$, $\tilde{u} = u/(3RT_s)^{\frac{1}{2}}$, $\tilde{T}_r = T_r/T_s$, $\tilde{T}_\theta = T_\theta/T_s$, $\tilde{T}_z = T_z/T_s$ and $\tilde{T} = T/T_s$, where r_s is the cylinder radius and ρ_s and T_s are surface values, to be defined later. Conservation of mass is simply

$$\tilde{\rho}\tilde{u}\, e^y = \dot{m}/\rho_s r_s(3RT_s)^{\frac{1}{2}}, \quad \text{a constant}, \tag{2.8}$$

and the remaining moment equations become

$$\frac{d\tilde{u}^2}{dy} = \frac{2}{3}\frac{M_f^2}{M_f^2-1}\left[\tilde{T}_\theta - \frac{\alpha\tilde{T}^\beta}{\tilde{u}^2}(\tilde{T}-\tilde{T}_r)\right], \tag{2.9a}$$

$$\frac{d\tilde{T}_r}{dy} = \frac{\alpha\tilde{T}^\beta}{\tilde{u}^2}(\tilde{T}-\tilde{T}_r) - \frac{2}{3}\left[\tilde{T}_\theta - \frac{\alpha\tilde{T}^\beta}{\tilde{u}^2}(\tilde{T}-\tilde{T}_r)\right]\Big/(M_f^2-1), \tag{2.9b}$$

$$\frac{d\tilde{T}_\theta}{dy} = \frac{\alpha\tilde{T}^\beta}{\tilde{u}^2}(\tilde{T}-\tilde{T}_\theta) - 2\tilde{T}_\theta, \tag{2.9c}$$

$$\frac{d\tilde{T}_z}{dy} = \frac{\alpha\tilde{T}^\beta}{\tilde{u}^2}(\tilde{T}-\tilde{T}_z), \tag{2.9d}$$

where $M_f^2 = \tilde{u}^2/\tilde{T}_r$, $\tilde{T} = \frac{1}{3}(\tilde{T}_r+\tilde{T}_\theta+\tilde{T}_z)$ and $\alpha = \dot{m}/3\mu_s$. Recognize that M_f *is not* the usual Mach number based on the equilibrium speed of sound, $a_e = (\frac{5}{3}RT)^{\frac{1}{2}}$. It is based on $a_f = (3RT_r)^{\frac{1}{2}}$, which might be thought of as a frozen speed of sound. The parameter α can be interpreted as a source Reynolds number or an inverse source Knudsen number.

In discussing the moment equations hereafter we shall deal only with the non-dimensional variables. Hopefully, it will cause no confusion if the tildes are dropped for notational simplicity.

As a check on the validity of the model, suppose $M_f^2 \gg 1$ as in previous analyses. Equation (2.9*a*) shows that $u^{-1}du/dy = O(M_f^{-2})$, or $u \sim u_\infty$, the terminal speed, and

$$\left.\begin{aligned} \frac{dT_r}{dy} &\sim \frac{\alpha T^\beta}{u_\infty^2}(T-T_r) \\ \frac{dT_\theta}{dy} &\sim \frac{\alpha T^\beta}{u_\infty^2}(T-T_\theta) - 2T_\theta \\ \frac{dT_z}{dy} &\sim \frac{\alpha T^\beta}{u_\infty^2}(T-T_z) \end{aligned}\right\} \quad \text{as} \quad M_f \to \infty. \tag{2.10}$$

It follows directly that $T_z \sim T_r$ and $T_\theta \sim 3T - 2T_r$ in the hypersonic far field, as expected in a cylindrical expansion. It is then possible to eliminate T_r by differentiation and find that the mean temperature satisfies

$$\frac{d^2T}{dy^2} + \left(2+\frac{\alpha T^\beta}{u_\infty^2}\right)\frac{dT}{dy} + \left(\frac{2}{3}\frac{\alpha T^\beta}{u_\infty^2}\right)T \sim 0 \quad \text{as} \quad M_f \to \infty. \tag{2.11}$$

This is the equation derived by Edwards & Cheng (1966), and an analytic solution can easily be constructed for Maxwellian molecules ($\beta = 0$). The treatment here reduces to the expected form in the hypersonic limit.

3. Asymptotic solution for large source Reynolds number

An asymptotic solution to the moment equations for large α can be developed using singular perturbation techniques. It will be convenient to replace y by u as the independent variable. Thus we consider

$$\frac{dT_r}{du} = \frac{(T_r - 3u^2)\,\alpha T^\beta u^{-2}(T-T_r) + 2T_r T_\theta}{u[\alpha T^\beta u^{-2}(T-T_r) - T_\theta]}, \tag{3.1a}$$

$$\frac{dT_\theta}{du} = \frac{3(T_r - u^2)[\alpha T^\beta u^{-2}(T-T_\theta) - 2T_\theta]}{u[\alpha T^\beta u^{-2}(T-T_r) - T_\theta]}, \tag{3.1b}$$

$$T + \tfrac{2}{3}T_r + u^2 = 4u_*^2, \quad \text{a constant}, \tag{3.1c}$$

$$T_z = 3T - (T_r + T_\theta). \tag{3.1d}$$

The third relation is simply the stagnation-enthalpy integral (2.4) rewritten in terms of non-dimensional variables. The convenience in writing the constant of integration as above will be obvious shortly.

First-order supersonic near-field solution

On the basis of continuum theory we expect the four temperatures to be equal to first order in the supersonic near field. With this assumption the stagnation-enthalpy integral gives

$$T_r = T_\theta = T_z = T = \tfrac{3}{5}(4u_*^2 - u^2). \tag{3.2}$$

Perturbation terms $O(\alpha^{-1})$ are now easily computed from (3.1), and they show that this solution breaks down as $u \to u_*$. We could anticipate this from the fact that $u = u_*$ and all temperatures equal to $\frac{9}{5}u_*^2$ corresponds to an equilibrium Mach number of one. So long as the temperatures are equal the flow is isentropic, according to (2.7), and a subsonic flow can only decelerate as it expands in the cylindrical geometry. Thus, since the flow generally starts subsonic and it must accelerate, a solution of the form (3.2) cannot be continued into the subsonic region $u < u_*$ near the cylinder surface.

The perturbation expansion also breaks down as $u \to 2u_*$, the terminal speed, unless $\beta = 0$. In this far-field regime the analysis leading to (2.11) is appropriate. Details of construction and matching are closely related to the discussion by Hamel & Willis (1966), so the arguments will not be repeated.

There is a tendency to identify the reference temperature T_s used in non-dimensionalization as the stagnation temperature of the gas stream. Then we should have $u_*^2 = \frac{5}{12}$ for a monatomic gas. In the treatment here, T_s is taken to be the temperature of the vaporizing material. Since these two temperatures are generally not the same, as will be seen later, we retain u_* as an undetermined constant.

First-order Knudsen-layer solution

The temperatures cannot be constrained to be nearly equal in the non-equilibrium subsonic regime. Equations valid in this Knudsen layer result by simply setting $\alpha = \infty$ in (3.1), giving

$$\frac{dT_r}{du} = \frac{T_r}{u} - 3u, \quad \frac{dT_\theta}{du} = \frac{(3T_r/u)(T-T_\theta)}{T-T_r}. \tag{3.3}$$

We can be a bit cavalier about solving (3.3) and matching with (3.2). Requiring all temperatures to be equal to $\frac{9}{5}u_*^2$ at $u = u_*$ and assuming $T_\theta = T_z$ at any point $u < u_*$, which eliminates the homogeneous solution for T_θ, leads to

$$\left.\begin{aligned} T_r &= \tfrac{24}{5}u_* u - 3u^2, \\ T_\theta = T_z &= 6u_*^2 - \tfrac{36}{5}u_* u + 3u^2, \\ T &= 4u_*^2 - \tfrac{16}{5}u_* u + u^2. \end{aligned}\right\} \tag{3.4}$$

A physical argument indicating we should expect $T_\theta = T_z$ at the surface is as follows. Suppose that the vaporizing material at the cylinder surface, be it liquid or solid, is in Maxwellian equilibrium, implying a single, well-defined surface temperature. A particle undergoing a phase change interacts with the surface work function, but that is expected to reduce only the energy flux normal to the surface. The lateral temperatures should be unaffected by the phase change, and hence they should remain unchanged and equal.

Intermediate transonic solution

A plot of the solutions (3.2) and (3.4) is given in figure 1. Note that there is nothing exceptional happening near $u/u_* = (\frac{3}{2})^{\frac{1}{2}}$, where $M_f = 1$; in effect, the frozen Mach number drops out of the problem at large source Reynolds number. Both function values and slopes match at the sonic point. Curvature mismatch implies that an intermediate expansion must be constructed for higher-order matching. We shall be interested in higher-order terms for comparison with the results of Edwards & Collins (1969) in § 4.

To construct the transonic expansion note that the first-order results in both the supersonic near field and the Knudsen layer reduce to

$$T_r, T_\theta, T_z, T \sim \tfrac{9}{5}u_*^2 - \tfrac{3}{5}(u^2 - u_*^2) + O(u^2 - u_*^2)^2 \quad \text{as} \quad u \to u_*.$$

Thus appropriate expansions should be of the form

$$\left.\begin{aligned} T_r &= \tfrac{9}{5}u_*^2 - \tfrac{3}{5}\alpha^{-\delta}v + \alpha^{-2\delta}T'_r, \\ T_\theta &= \tfrac{9}{2}u_*^2 - \tfrac{3}{5}\alpha^{-\delta}v + \alpha^{-2\delta}T'_\theta, \\ T &= \tfrac{9}{5}u_*^2 - \tfrac{3}{5}\alpha^{-\delta}v + \alpha^{-2\delta}T', \end{aligned}\right\} \tag{3.5}$$

where $v = \alpha^\delta(u^2 - u_*^2)$ and δ is to be chosen.

Take $4u_*^2$ as the *exact* constant in the stagnation-enthalpy integral, which is to be expanded later as a function of α. Then the integral gives $T' = -\frac{2}{3}T'_r$. Equation (3.1) for T_r becomes

$$dT'_r/dv = -\tfrac{3}{5}[2v/u_*^2 + \tfrac{6}{5}u_*^2(\tfrac{9}{5}u_*^2)^{1-\beta}/T'_r] \tag{3.6}$$

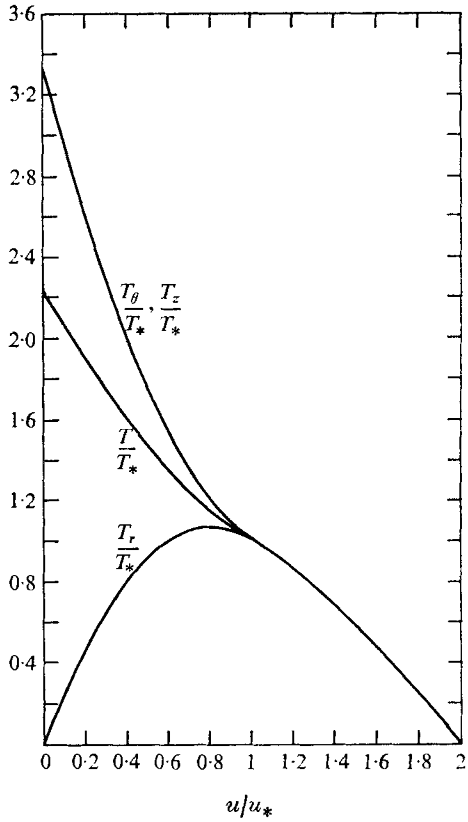

FIGURE 1. First-order flow properties for large source Reynolds numbers, normalized with respect to values at the sonic point.

provided $\delta = \frac{1}{3}$. Expansion of (3.2) and (3.4) gives the matching conditions as $T'_r \sim -3v^2/5u_*^2$ as $v \to -\infty$ and $T'_r \to 0$ as $v \to +\infty$.

The transformation $T'_r = -\frac{3}{5}(v/u_*)^2 + Ax$ changes (3.6) into a Riccati equation for v. Thus set $v = BF^{-1}dF/dx$ and choose the constants as

$$A = (\tfrac{5}{3}u_*^2)^{\frac{1}{3}}[\tfrac{18}{25}u_*^2(\tfrac{9}{5}u_*^2)^{1-\beta}]^{\frac{2}{3}}, \quad B = -(\tfrac{5}{3}u_*^2)^{\frac{2}{3}}[\tfrac{18}{25}u_*^2(\tfrac{9}{5}u_*^2)^{1-\beta}]^{\frac{1}{3}} \tag{3.7}$$

to arrive at the Airy equation $d^2F/dx^2 = xF$.

With A positive we shall evidently have to have $x \to \infty$ as $v \to \infty$. Recall that the Airy function $\mathrm{Bi}(x)$ is exponentially large compared with $\mathrm{Ai}(x)$ as $x \to \infty$ and that Bi has the same sign as its derivative for $x > 0$. Therefore, with B negative, F must be equal to $\mathrm{Ai}(x)$ within a numerical factor which is irrelevant. Asymptotic expansions for $\mathrm{Ai}(x)$ and its derivative lead to

$$\left.\begin{aligned} x &\sim (v/B)^2 - B/2v \\ T'_r &\sim \left(\frac{A}{B^2} - \frac{3}{5u_*^2}\right)v^2 - \frac{AB}{2v} \end{aligned}\right\} \quad \text{as} \quad v \to \infty. \tag{3.8}$$

As it happens, $B^2 = \frac{5}{3}u_*^2 A$ and the boundary condition required by matching as $v \to \infty$ is satisfied.

Ai(x) and its derivative are always of opposite sign in the range $0 \leqslant x \leqslant \infty$. To get the branch $v < 0$ we must consider $x < 0$ and x lying in the range up to the first zero of Ai(x) at $x_0 = -2{\cdot}338$. In the vicinity of x_0,

$$\mathrm{Ai}(x) \sim (x-x_0)\,\mathrm{Ai}\,(x_0) + O(x-x_0)^2$$

and so

$$\left.\begin{aligned} x &\sim x_0 + B/v \\ T'_r &\sim -\tfrac{3}{5}(v/u_*)^2 + Ax_0 + AB/v \end{aligned}\right\} \quad \text{as} \quad v \to -\infty. \tag{3.9}$$

The constant Ax_0 is necessary and requires a term $O(\alpha^{-\frac{2}{3}})$ in the Knudsen-layer expansion for matching to follow. We shall return to constructing the appropriate term shortly.

As an interesting aside, there is a similarity between the preceding construction and that of Edwards & Collins (1969). This is somewhat surprising since their starting point, the Navier–Stokes equations, seems quite different.

Substitution of the transonic expansions into equations (3.1) for T_θ leads to $T'_\theta = -\frac{3}{2}T'_r$. There will be no problem in matching with the supersonic near field since the perturbations there have this relationship as $u \to u_*$. Matching with the Knudsen layer can be accomplished if there is a term in the Knudsen-layer expansion for T_θ approaching $-\frac{3}{2}Ax_0\alpha^{-\frac{2}{3}}$ as the sonic point is approached.

Second-order terms in the Knudsen layer

Add to (3.4) perturbation terms of $\alpha^{-\frac{2}{3}}$ times T'_r, T'_θ, T'_z and T', respectively. The stagnation-enthalpy integral gives $T' = -\frac{2}{3}T'_r$ and (3.1) for T_r leads to

$$dT'_r/du = T'_r/u.$$

The solution matching the transonic result is $T'_r = Ax_0 u/u_*$. The equation for T'_θ is

$$dT'_\theta/du = -\tfrac{3}{2}T'_r/u - 3(\tfrac{6}{5}u_* - u)\,(T'_\theta + \tfrac{3}{2}T'_r)/(u-u_*)^2.$$

A particular solution is evidently $T'_\theta = -\frac{3}{2}T'_r$ and the homogeneous solution is easily constructed. However, the latter is eliminated if we require $T_\theta = T_z$ at any point $u < u_*$, which will be done. It follows that the complete expansions to this order in the Knudsen layer are

$$\left.\begin{aligned} T_r &= \tfrac{24}{5}u_* u - 3u^2 + \alpha^{-\frac{2}{3}}Ax_0 u/u_*, \\ T_\theta = T_z &= 6u_*^2 - \tfrac{36}{5}u_* u + 3u^2 - \tfrac{3}{2}\alpha^{-\frac{2}{3}}Ax_0 u/u_*, \\ T &= 4u_*^2 - \tfrac{16}{5}u_* u + u^2 - \tfrac{2}{3}\alpha^{-\frac{2}{3}}Ax_0 u/u_*. \end{aligned}\right\} \tag{3.10}$$

The next term in the Knudsen-layer expansion is a term of order $\alpha^{-1}\ln\alpha$. We shall not concern ourselves with this further development here.

4. An *ad hoc* surface evaporation model

Surface boundary conditions

To complete the description we need a set of boundary conditions at the cylinder surface. In deriving these we revert to dimensional variables. The approach taken here follows Anisimov (1968) in assuming that phase equilibrium exists

$u_* = 0{\cdot}59$	$u_0 = 0{\cdot}34$	$T_{r0} = 0{\cdot}63$
$T_{\theta 0} = T_{z0} = 1$	$T_0 = 0{\cdot}88$	$\rho = 0{\cdot}56$
$\epsilon = 1{\cdot}12$	$T_t = 0{\cdot}85$	$\phi = 0{\cdot}84$

TABLE 1

just inside the surface. Then we can assume that the distribution function for emitted particles has the form

$$f = \rho_s(2\pi RT_s)^{-\frac{3}{2}} \exp\left[-\frac{\xi^2+\eta^2+\zeta^2}{2RT_s}\right] \quad \text{for} \quad \xi > 0, \tag{4.1a}$$

where ρ_s is the saturated vapour density at the surface temperature T_s. A non-zero distribution function is needed for $\xi < 0$ to account for backscattering. There is no choice obviously better than assuming that it is proportional to the ellipsoidal approximant for negative ξ. Thus

$$f = \epsilon\rho(2\pi R)^{-\frac{3}{2}}(T_r T_\theta T_z)^{-\frac{1}{2}} \exp\left[-\frac{(\xi-u)^2}{2RT_r} - \frac{\eta^2}{2RT_\theta} - \frac{\zeta^2}{2RT_z}\right] \quad \text{for} \quad \xi < 0. \tag{4.1b}$$

The parameter ϵ can be thought of as a sticking coefficient times the ratio of density of backscattered particles to gas density (ρ) at the surface.

Boundary conditions for the five parameters in the ellipsoidal approximant follow by requiring appropriate moments of (1.1) to match those of (4.1) at the surface ($y = 0$). The BGK model (2.1) implies that these moments should be taken with respect to ξ, ξ^2, ξ^3, $\xi\eta^2$ and $\xi\zeta^2$. After considerable manipulation this leads to the following results stated in terms of the non-dimensional variables $\tilde{\rho} = \rho/\rho_s$, $\tilde{u} = u/(3RT_s)^{\frac{1}{2}}$, $\tilde{T}_r = T_r/T_s$, $\tilde{T}_\theta = T_\theta/T_s$ and $\tilde{T}_z = T_z/T_s$:

$$\tilde{T}_\theta = \tilde{T}_z = 1, \quad \tilde{T}_r = 1-(\tfrac{3}{8}\pi)^{\frac{1}{2}}\tilde{u}, \tag{4.2a,b}$$

$$\tilde{\rho} = \{[1-\pi^{\frac{1}{2}}m\exp(m^2)\operatorname{erfc}(m)] + \tilde{T}_r^{\frac{1}{2}}[(2m^2+1)\exp(m^2)\operatorname{erfc}(m) - 2m\pi^{-\frac{1}{2}}]\}/2\tilde{T}_r, \tag{4.2c}$$

$$\epsilon = [(2m^2+1) - m(\pi/\tilde{T}_r)^{\frac{1}{2}}]\, e^{m^2}/\tilde{\rho}\tilde{T}_r^{\frac{1}{2}}, \tag{4.2d}$$

with $m = \tilde{u}/(\tfrac{2}{3}\tilde{T}_r)^{\frac{1}{2}}$. All values in (4.2) are to be evaluated at the surface. Below the tildes will be dropped with the understanding that all variables are non-dimensional.

Comparison with previous work

As a check on the surface boundary conditions to be used here, consider the case of large source Reynolds number for comparison with other results in the literature. The first-order results given in table 1 were obtained by combining (3.4) with (4.2) and solving numerically. The parameter T_t is the gas stagnation temperature divided by the surface temperature, and ϕ is the evaporation rate normalized with respect to the value without backscattering. That is,

$$T_t = \tfrac{12}{5}u_*^2, \quad \phi = \dot{m}/\rho_s r_s (RT_s/2\pi)^{\frac{1}{2}}, \tag{4.3}$$

with $\dot{m}$ computed from (2.8) for $y = 0$.

	$\beta = 0$	$\beta = 0{\cdot}25$	$\beta = 0{\cdot}50$
u'_*	$-0{\cdot}0162$	$-0{\cdot}0175$	$-0{\cdot}0189$
u'_0	$0{\cdot}194$	$0{\cdot}210$	$0{\cdot}226$
T'_{r0}	$-0{\cdot}211$	$-0{\cdot}228$	$-0{\cdot}245$
T'_0	$-0{\cdot}0703$	$-0{\cdot}0758$	$-0{\cdot}0818$
ρ'	$-0{\cdot}187$	$-0{\cdot}202$	$-0{\cdot}217$
ϵ'	$0{\cdot}302$	$0{\cdot}325$	$0{\cdot}351$
T'_t	$-0{\cdot}0462$	$-0{\cdot}0499$	$-0{\cdot}0538$
ϕ'	$0{\cdot}193$	$0{\cdot}208$	$0{\cdot}225$

TABLE 2

Edwards & Collins (1969) give $T_t = 0{\cdot}90$ and $\phi = 0{\cdot}81$ and Anisimov (1968) gives $T_t = 0{\cdot}89$ and $\phi = 0{\cdot}82$. Although these treatments assume spherical and slab symmetry, respectively, a comparison is valid because geometric effects are $O(\alpha^{-1})$ for large α. Note the stagnation-temperature jump at the surface. Presumably, part of the heat required for change of phase is being supplied by the expanding vapour. The significantly lower value of T_t reported here may be due to the altered form of the stagnation-enthalpy integral (2.4). Both the other treatments assume that a single temperature characterizes the gas flow.

Second-order results are proportional to $\alpha^{-\frac{2}{3}}$ and can be obtained by combining (3.10) with (4.2) and determining the rate of change of all variables with respect to $\alpha^{-\frac{2}{3}}$ by numerical differentiation. Setting $u_* = \overline{u}_* + \alpha^{-\frac{2}{3}} u'_*$, etc., the perturbation coefficients are as given in table 2 for three representative values of β. The value of ϕ' implied by Edwards & Collins (1969) is $1{\cdot}14$. They assume constant viscosity, or $\beta = 0$. No explanation of the disagreement is available.

5. Numerical solution for moderate source Reynolds number

Formulation as a boundary-value problem

For moderate source Reynolds numbers it is necessary to solve (2.9) numerically. Since y does not appear explicitly, it is convenient to introduce u^2 as the independent variable to reduce the order of the differential system. Thus we shall be considering

$$\frac{dT_r}{du^2} = \frac{\alpha T^\beta (T_r - 3u^2)(T - T_r) + 2u^2 T_r T_\theta}{2u^2[\alpha T^\beta (T - T_r) - u^2 T_\theta]}, \tag{5.1a}$$

$$\frac{dT_\theta}{du^2} = \frac{3(T_r - u^2)[\alpha T^\beta (T - T_\theta) - 2u^2 T_\theta]}{2u^2[\alpha T^\beta (T - T_r) - u^2 T_\theta]}, \tag{5.1b}$$

$$\frac{dT_z}{du^2} = \frac{3\alpha T^\beta (T_r - u^2)(T - T_z)}{2u^2[\alpha T^\beta (T - T_r) - u^2 T_\theta]}. \tag{5.1c}$$

The relationship between u^2 and y is left until later.

Let us turn now to boundary conditions to be imposed in solving the first-order system (5.1). The cylinder surface is the most obvious boundary point. Boundary conditions there are given in (4.2). A consequence of the choice of independent variable is that the 'location' of this boundary point is not known *a priori*. That is, the value of u^2 at the surface must be determined as part of the solution procedure. This entails no great difficulty.

In general the differential system is poorly behaved near $M_f = 1$, as can be seen most clearly by returning to the equation for u^2 in (2.9). The only way to achieve monotonically increasing velocity (as y increases) is to have the following conditions holding simultaneously at this critical point:

$$T_r = u^2, \quad \alpha T^\beta (T - T_r) = u^2 T_\theta. \tag{5.2}$$

The first requirement is simply $M_f = 1$ and the second is a compatibility condition which causes the right-hand sides of (5.1) to be indeterminate forms. The critical state can be characterized as a saddle point. Again the value of u^2 at this second boundary point, and hence its 'location', is not known *a priori*.

Imposing (5.2) can be thought of as a physical requirement which guarantees that the solutions obtained will be meaningful. The situation is analogous to one-dimensional isentropic continuum flow, where steady sonic flow can be maintained only at a section of minimum area. Further discussion of critical points in ordinary one-dimensional flows is given by Shapiro (1953).

There are now five boundary conditions, which should be adequate to determine three constants of integration and the values of u^2 at the two boundary points. Formulation as a two-point boundary-value problem is perfectly proper. However, it seems preferable to continue integration to a third point well into the region $M_f > 1$. A numerical scheme for three-point boundary-value problems will be used here, with integration continuing to $u^2 = 1$. The choice of 'one' as the third boundary point is essentially arbitrary, and no boundary conditions are to be imposed at it. Beyond $u^2 = 1$ the system (2.9) can be integrated straightforwardly.

The numerical procedure

A quasi-linearization procedure will be used to solve the three-point boundary-value problem. It can be described as follows. Consider a first-order ordinary differential system of the form

$$d\phi/dx = F(x, \phi), \tag{5.3}$$

where ϕ is a vector having N_v components. In general there can be N_a nonlinear algebraic conditions on ϕ at $x = x_a$, N_c conditions at $x = x_c$ and N_b conditions at $x = x_b$. It is most convenient to assume that these conditions involve values of ϕ only at the boundary point in question. The locations of x_a and x_c are to be determined, so for a closed system we require

$$N_a + N_c + N_b = N_v + 2. \tag{5.4}$$

To facilitate imposing boundary conditions the nodal system $\{x_j\}$ should always contain the boundary points as nodes. To this end, set

$$x_j = \frac{(M-j)(N-j)}{(M-1)(N-1)} x_a + \frac{(j-1)(N-j)}{(M-1)(N-M)} x_c + \frac{(j-1)(j-M)}{(N-1)(N-M)} x_b, \quad 1 \leqslant j \leqslant N, \tag{5.5}$$

with $1 < M < N$. Note that the nodal locations will change as the values of x_a and x_c are iteratively determined.

On this nodal system (5.3) is approximated by a central finite-difference scheme:

$$\phi_{j+1} = \phi_j + (x_{j+1} - x_j)\, F(\tfrac{1}{2}(x_{j+1} + x_j), \tfrac{1}{2}(\phi_{j+1} + \phi_j)). \tag{5.6}$$

This gives $(N-1)\,N_v$ nonlinear algebraic equations. These together with the boundary conditions give a system of $NN_v + 2$ equations in the same number of variables. In a quasi-linearization procedure the system is solved by Newton–Raphson iteration. Full advantage is taken of the special structure of the Jacobian matrix, whose elements can be evaluated by numerical differentiation.

Typically, about 150 equations are involved and convergence is rapid with a good initial guess. Generating an initial guess was found to be critical. This was done by starting at $\alpha = 1000$, where the asymptotic forms of § 3 can be used, and sequentially reducing α until the range of interest was covered. The previous solution was used as an initial guess for each new value of α. About 20 steps were required for the results reported.

Numerical results

Solutions for Maxwellian molecules ($\beta = 0$) and several source Reynolds numbers are given in figure 2. In figure 2 (*a*) note that the surface and critical values of u^2 are converging as α decreases. For $\alpha < 10$ the numerically determined surface values of M_f are reasonably well fitted by

$$M_{f0} \approx 0{\cdot}494 + 0{\cdot}437/\alpha, \quad \alpha < 10. \tag{5.7}$$

This, together with (4.2), allows the locus of surface conditions to be approximately extended as shown by the curved dashed line.

Similar results for hard-sphere molecules ($\beta = \frac{1}{2}$) are given in figure 3. The effect of β in the Knudsen layer and the supersonic near field is not large.

Parameter values at the surface ($y = 0$) for these two cases are given in tables 3 and 4. The value of ϕ, the mass-flux parameter defined in (4.3), is expected to approach one in the free molecular limit. As α decreases, the flow becomes increasingly rarefied and we see that ϕ is indeed increasing, presumably towards one. Smaller values of α were not attempted owing to difficulty in achieving convergence of the Newton–Raphson iteration.

Flow properties for Maxwellian molecules and $\alpha = 10$ are plotted versus the co-ordinate $y = \ln(r/r_s)$ in figures 4 and 5. These results were obtained by integrating (5.1) to $u^2 = 1$ and determining associated y values from

$$\frac{dy}{du^2} = \frac{3}{2}\frac{T_r - u^2}{\alpha T^\beta (T - T_r) - u^2 T_\theta}. \tag{5.8}$$

Beyond $u^2 = 1$, (2.9) are integrated using a fourth-order Runge–Kutta scheme. The entropy variable

$$(s - s_0)/R = \ln[\rho^{-1}(T_r T_\theta T_z)^{\frac{1}{2}}], \tag{5.9}$$

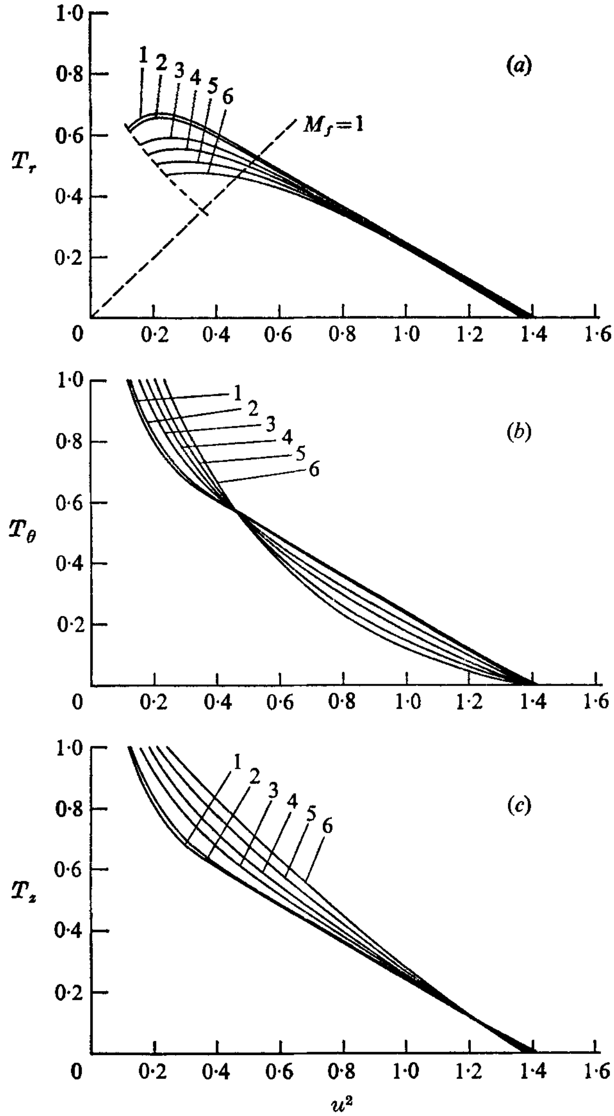

FIGURE 2. Orthogonal temperature variation with u^2 for $\beta = 0$. Curves: 1, $\alpha = 1000$; 2, $\alpha = 100$; 3, $\alpha = 10$; 4, $\alpha = 5$; 5, $\alpha = 3$; 6, $\alpha = 2$.

stated in terms of *non-dimensional* density and temperatures, follows from (2.6) with

$$\frac{s_0}{R} = \tfrac{5}{2} + \ln\left[\frac{m^4}{\rho_s h^3}(2\pi R T_s)^{\frac{3}{2}}\right]. \tag{5.10}$$

Since $ds/dy > 0$ for all y, the flow is never in translational equilibrium for $\alpha = 10$.

Note that entropy increases linearly with y in figure 4 as u approaches its terminal value. This is a result which follows only for Maxwellian molecules.

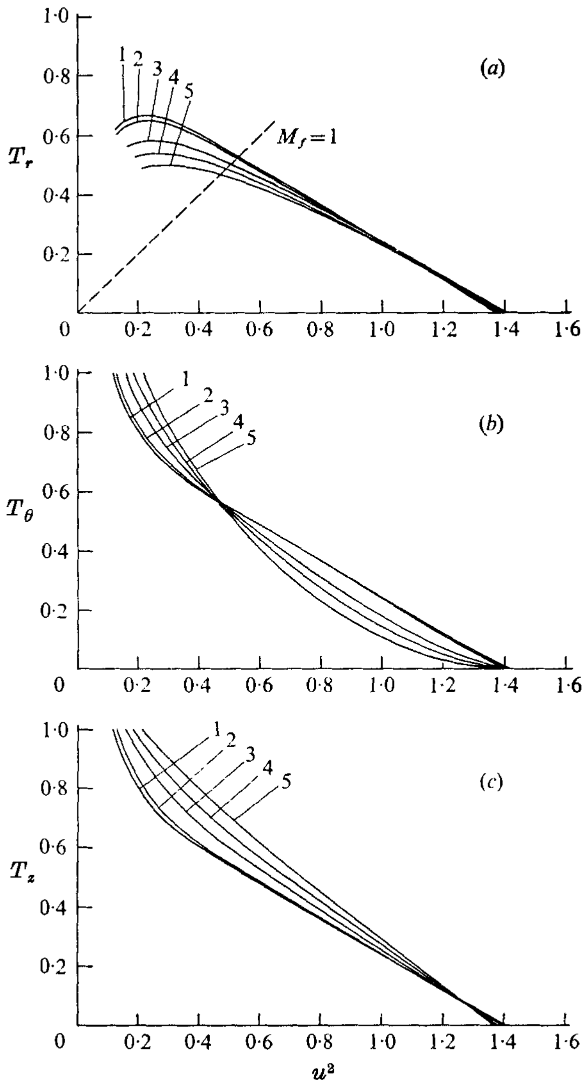

FIGURE 3. Orthogonal temperature variation with u^2 for $\beta = 0{\cdot}5$. Curves: 1, $\alpha = 1000$; 2, $\alpha = 100$; 3, $\alpha = 10$; 4, $\alpha = 5$; 5, $\alpha = 3$.

α	ρ_0	u_0	T_{r0}	ϕ	T_t
1000	0·558	0·346	0·624	0·840	0·846
100	0·550	0·355	0·615	0·848	0·844
10	0·513	0·395	0·571	0·881	0·836
5	0·488	0·425	0·539	0·899	0·832
3	0·462	0·456	0·505	0·914	0·828
2	0·436	0·488	0·470	0·925	0·825

TABLE 3

α	ρ_0	u_0	T_{r0}	ϕ	T_t
1000	0·558	0·347	0·624	0·840	0·846
100	0·549	0·356	0·613	0·849	0·844
10	0·508	0·401	0·565	0·884	0·835
5	0·481	0·433	0·530	0·904	0·831
3	0·453	0·467	0·493	0·918	0·827

TABLE 4

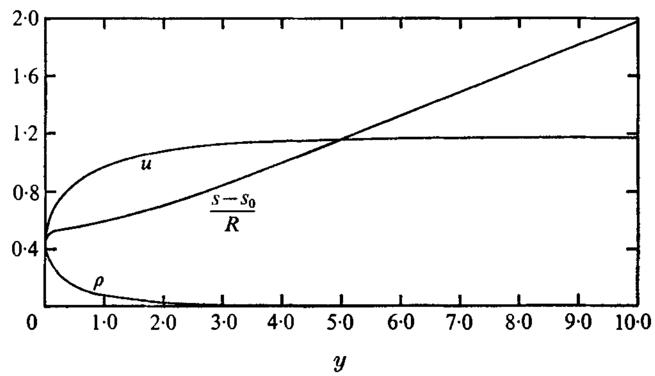

FIGURE 4. Density, mean velocity and entropy *vs.* $y = \ln(r/r_s)$ for $\beta = 0$ and $\alpha = 10$.

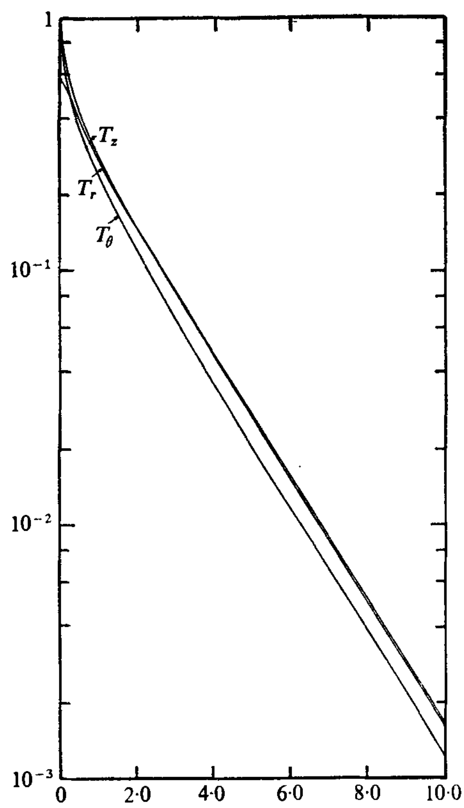

FIGURE 5. Orthogonal temperatures *vs.* $y = \ln(r/r_s)$ for $\beta = 0$ and $\alpha = 10$.

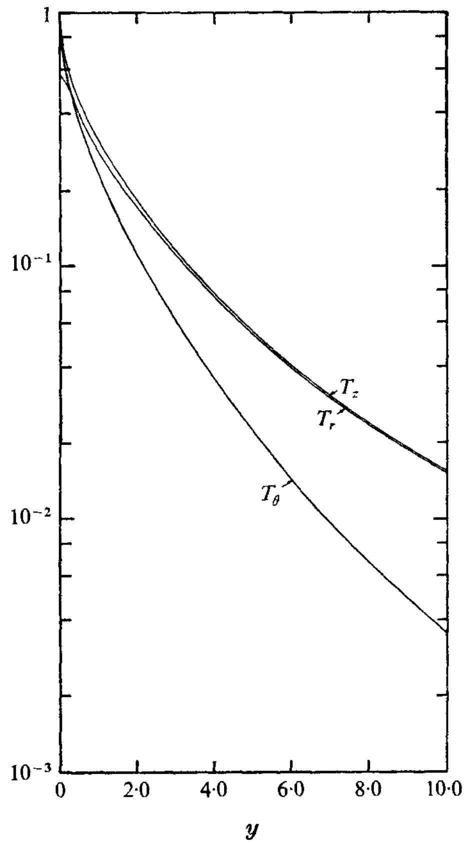

FIGURE 6. Orthogonal temperatures *vs.* $y = \ln(r/r_s)$ for $\beta = 0{\cdot}5$ and $\alpha = 10$.

Equation (2.11) and the analysis preceding it imply that all temperatures are proportional to $e^{-\lambda y}$ for $\beta = 0$, with

$$\lambda = 1 + \frac{\alpha}{8u_*^2} - \left[\left(1 + \frac{\alpha}{8u_*^2}\right)^2 - \frac{\alpha}{6u_*^2}\right]^{\frac{1}{2}}. \tag{5.11}$$

This dependence on y is clearly evident in figure 5. Also, according to (2.8), density becomes proportional to e^{-y} as u approaches its asymptotic value of $2u_*$. Thus

$$(s - s_0)/R \sim \text{constant} + (1 - \tfrac{3}{2}\lambda)\, y \quad \text{as} \quad y \to \infty \quad \text{for} \quad \beta = 0. \tag{5.12}$$

The value of λ is 0·555 for $\alpha = 10$, implying slopes in agreement with those shown in figures 4 and 5.

For $\beta \neq 0$ the trend in (5.12) does not hold because the temperatures in the far field do not depend exponentially on y. The dependence for hard-sphere molecules can be seen in figure 6. The explicit form follows from (2.10):

$$\tfrac{3}{2}T_r,\ \tfrac{3}{2}T_z,\ T \sim [(\alpha\beta/12u_*^2)(y - y_0)]^{-1/\beta}, \tag{5.13a}$$

$$T_\theta \sim \frac{\alpha}{8u_*^2}\left[\frac{\alpha\beta}{12u_*^2}(y - y_0)\right]^{-(1+1/\beta)}. \tag{5.13b}$$

The representation breaks down as β approaches zero.

6. Concluding remarks

The approach used here is recognized as *ad hoc* in using the BGK equation, in assuming an ellipsoidal approximant for the distribution function, and in the treatment of boundary conditions at the surface of the cylindrical source. Its main merit is that it provides a generalized treatment of evaporation into a surrounding vacuum which may be related to previous solutions in appropriate limits. Some conclusions which have been drawn follow.

(*a*) A mechanism causing the gas to accelerate and become supersonic can be postulated. Interaction of escaping atoms with the surface work function causes them to belong to a highly non-Maxwellian distribution after the phase change. Collisions in the gas phase relax the distribution function towards a Maxwellian form. This implies increasing entropy and consequently the subsonic flow tends to choke. In effect, the flow loses the information that it started subsonic. Once the flow is choked, stream-tube divergence in a cylindrical geometry continues to accelerate the flow to terminal speed.

(*b*) For the model discussed here there is a generalized stagnation-enthalpy integral. The associated stagnation temperature is found to be lower than the temperature of the liquid (or solid) at the cylindrical phase interface. A plausible explanation for this is that a portion of the latent heat of vaporization required for phase change is being supplied by the expanding gas flow.

(*c*) For moderate values of the source Reynolds number α, there is a strong dependence of flow properties on α. Numerical results show that, as α decreases, the stagnation temperature of the gas stream decreases and the mass flow increases towards its free molecular value. On the other hand, for large values of α the flow becomes independent of α to first order and the results of the model agree to this order with previous work. Owing to the results of Cattolica *et al.* (1974) it is doubtful whether the particular model advanced here is valid for small values of α. No attempt has been made to assess the limits of validity of the model.

(*d*) The variation of flow properties with β, which characterizes the dependence of collision frequency on temperature, is relatively weak in the Knudsen layer and supersonic near field. This is true, in particular, for the stagnation temperature of the gas stream and the mass flow. In the hypersonic far field the behaviour of the three orthogonal temperatures used in the model depends strongly on β and is similar to early treatments by Edwards & Cheng (1966) and Hamel & Willis (1966). The only new feature is a scale dependence on α introduced by an altered solution in the supersonic near field. This is particularly evident in the curves for T_θ in figures 2 and 3.

(*e*) Numerical solutions at moderate source Reynolds number ($\alpha = 10$) show that the flow field never achieves a state of translational equilibrium. In the Knudsen layer entropy production is intense owing to relaxation of the non-Maxwellian surface distribution function. Later the collision frequency, which is proportional to α, is too small to maintain equilibrium as the flow state changes owing to the expansion process. This observation is in marked contrast to the case of large source Reynolds number, where the supersonic near field is in translational equilibrium to within terms $O(\alpha^{-1})$.

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